CHAPTER III

TURBULENCE

§26. Stability of steady flow

For any problem of viscous flow under given steady conditions there must in principle exist an exact steady solution of the equations of fluid dynamics. These solutions formally exist for all Reynolds numbers. Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. Those which do must not only obey the equations of fluid dynamics, but also be stable. Any small perturbations which arise must decrease in the course of time. If, on the contrary, the small perturbations which inevitably occur in the flow tend to increase with time, the flow is unstable and cannot actually exist.†

The mathematical investigation of the stability of a given flow with respect to infinitely small perturbations will proceed as follows. On the steady solution concerned (whose velocity distribution is $\mathbf{v}_0(\mathbf{r})$, say), we superpose a non-steady small perturbation $\mathbf{v}_1(\mathbf{r}, t)$, which must be such that the resulting velocity $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ satisfies the equations of motion. The equation for $\mathbf{v}_1$ is obtained by substituting in the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\frac{\mathbf{grad}\, p}{\rho} + \nu \triangle \mathbf{v}, \qquad \operatorname{div} \mathbf{v} = 0 \tag{26.1}$$

the velocity and pressure

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \qquad p = p_0 + p_1, \tag{26.2}$$

where the known functions $\mathbf{v}_0$ and p_0 satisfy the unperturbed equations

$$(\mathbf{v}_0 \cdot \mathbf{grad})\mathbf{v}_0 = -\frac{\mathbf{grad}\, p_0}{\rho} + \nu \triangle \mathbf{v}_0, \qquad \operatorname{div} \mathbf{v}_0 = 0. \tag{26.3}$$

Omitting terms above the first order in $\mathbf{v}_1$, we obtain

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \mathbf{grad})\mathbf{v}_1 + (\mathbf{v}_1 \cdot \mathbf{grad})\mathbf{v}_0 = -\frac{\mathbf{grad}\, p_1}{\rho} + \nu \triangle \mathbf{v}_1, \qquad \operatorname{div} \mathbf{v}_1 = 0. \tag{26.4}$$

The boundary condition is that $\mathbf{v}_1$ vanish on fixed solid surfaces.

Thus $\mathbf{v}_1$ satisfies a system of homogeneous linear differential equations, with coefficients that are functions of the coordinates only, and not of the time. The general solution of such equations can be represented as a sum of particular solutions in which $\mathbf{v}_1$ depends on time

† In the previous edition, instability with respect to infinitesimal perturbations was called *absolute instability*. This adjective will not now be used in the present context, but will serve (in accordance with more customary terminology) as a contrast to *convected* (§28).

as $e^{-i\omega t}$. The frequencies ω of the perturbations are not arbitrary, but are determined by solving the equations (26.4) with the appropriate boundary conditions. The frequencies are in general complex. If there are ω whose imaginary parts are positive, $e^{-i\omega t}$ will increase indefinitely with time. In other words, such perturbations, once having arisen, will increase, i.e. the flow is unstable with respect to such perturbations. For the flow to be stable it is necessary that the imaginary part of any possible frequency ω be negative. The perturbations that arise will then decrease exponentially with time.

Such a mathematical investigation of stability is extremely complicated, however. The theoretical problem of the stability of steady flow past bodies with finite dimensions has not yet been solved. It is certain that steady flow is stable for sufficiently small Reynolds numbers. The experimental data seem to indicate that, when R increases, it eventually reaches a value R_{cr} (the *critical Reynolds number*) beyond which the flow is unstable with respect to infinitesimal disturbances. For sufficiently large Reynolds numbers ($R > R_{cr}$), steady flow past solid bodies is therefore impossible. The critical Reynolds number is not, of course, a universal constant, but takes a different value for each type of flow. These values appear to be of the order of 10 to 100; for example, in flow across a cylinder undamped non-steady flow has been observed for $R = ud/\nu \simeq 30$, d being the diameter of the cylinder.

Let us now consider the nature of the non-steady flow which is established as a result of the instability of steady flow at large Reynolds numbers (L. D. Landau 1944). We begin by examining the properties of this flow at Reynolds numbers only slightly greater than R_{cr}. For $R < R_{cr}$ the imaginary parts of the complex frequencies $\omega = \omega_1 + i\gamma_1$ for all possible small perturbations are negative ($\gamma_1 < 0$). For $R = R_{cr}$ there is one frequency whose imaginary part is zero. For $R > R_{cr}$ the imaginary part of this frequency is positive, but, when R is close to R_{cr}, γ_1 is small in comparison with the real part ω_1.† The function $\mathbf{v}_1$ corresponding to this frequency is of the form

$$\mathbf{v}_1 = A(t)\mathbf{f}(x, y, z), \tag{26.5}$$

where $\mathbf{f}$ is some complex function of the coordinates, and the complex amplitude $A(t)$ is‡

$$A(t) = \text{constant} \times e^{\gamma_1 t}e^{-i\omega_1 t}. \tag{26.6}$$

This expression for $A(t)$ is actually valid, however, only during a short interval of time after the disruption of the steady flow; the factor $e^{\gamma_1 t}$ increases rapidly with time, whereas the method of determining $\mathbf{v}_1$ given above, which leads to expressions like (26.5) and (26.6), applies only when $|\mathbf{v}_1|$ is small. In reality, of course, the modulus $|A|$ of the amplitude of the non-steady flow does not increase without limit, but tends to a finite value. For R close to R_{cr} (we always mean, of course, $R > R_{cr}$), this finite value is small, and can be determined as follows.

Let us find the time derivative of the squared amplitude $|A|^2$. For very small values of t, when (26.6) is still valid, we have $\mathrm{d}|A|^2/\mathrm{d}t = 2\gamma_1|A|^2$. This expression is really just the first term in an expansion in series of powers of A and A^*. As the modulus $|A|$ increases (still remaining small), subsequent terms in this expansion must be taken into account. The

† The set (or *spectrum*) of all possible perturbation frequencies for a given type of flow includes both separate isolated values (the *discrete spectrum*) and the whole of various frequency ranges (the *continuous spectrum*). It seems that for flow past finite bodies the frequencies with $\gamma_1 > 0$ can occur only in the discrete spectrum. The reason is that the perturbations corresponding to the frequencies in the continuous spectrum are in general not zero at infinity, but the unperturbed flow there is certainly a stable homogeneous plane-parallel flow.

‡ As usual, we understand the real part of (26.6).

next terms are those of the third order in A. However, we are not interested in the exact value of the derivative $\mathrm{d}\,|A|^2/\mathrm{d}t$, but in its time average, taken over times large compared with the period $2\pi/\omega_1$ of the factor $e^{-i\omega_1 t}$; we recall that, since $\omega_1 \gg \gamma_1$, this period is small compared with the time $1/\gamma_1$ required for the amplitude modulus $|A|$ to change appreciably. The third-order terms, however, must contain the periodic factor, and therefore vanish on averaging.† The fourth-order terms include one which is proportional to $A^2 A^{*2} = |A|^4$ and which does not vanish on averaging. Thus we have as far as fourth-order terms

$$\overline{\mathrm{d}\,|A|^2/\mathrm{d}t} = 2\gamma_1\,|A|^2 - \alpha\,|A|^4, \tag{26.7}$$

where α (the *Landau constant*) may be either positive or negative.

We are interested in the case where an infinitesimal perturbation (superimposed on the original flow) first becomes unstable for $R > R_{cr}$. This corresponds to $\alpha > 0$. We have not put bars above $|A|^2$ and $|A|^4$ in (26.7), since the averaging is only over time intervals short compared with $1/\gamma_1$. For the same reason, in solving the equation we proceed as if the bar were omitted above the derivative also. The solution of equation (26.7) is

$$1/|A|^2 = \alpha/2\gamma_1 + \text{constant} \times e^{-2\gamma_1 t}.$$

Hence it is clear that $|A|^2$ tends asymptotically to a finite limit:

$$|A|^2_{\max} = 2\gamma_1/\alpha. \tag{26.8}$$

The quantity γ_1 is some function of the Reynolds number. Near R_{cr} it can be expanded as a series of powers of $\mathrm{R} - \mathrm{R}_{cr}$. But $\gamma_1(\mathrm{R}_{cr}) = 0$, by the definition of the critical Reynolds number. Hence we have to the first order

$$\gamma_1 = \text{constant} \times (\mathrm{R} - \mathrm{R}_{cr}). \tag{26.9}$$

Substituting this in (26.8), we see that the modulus $|A|$ of the amplitude is proportional to the square root of $\mathrm{R} - \mathrm{R}_{cr}$:

$$|A|_{\max} \propto \sqrt{(\mathrm{R} - \mathrm{R}_{cr})}. \tag{26.10}$$

Let us now briefly discuss the case where $\alpha < 0$ in (26.7). The two terms in that expansion are then insufficient to determine the limiting amplitude of the perturbation, and we have to include a negative term of higher order; let this be $-\beta|A|^6$ with $\beta > 0$, which gives

$$|A|^2_{\max} = \frac{|\alpha|}{2\beta} \pm \sqrt{\left(\frac{\alpha^2}{4\beta^2} + \frac{2|\alpha|}{\beta}\gamma_1\right)}, \tag{26.11}$$

with γ_1 as in (26.9). The dependence is shown in Fig. 13b; Fig. 13a corresponds to $\alpha > 0$, (26.10). When $\mathrm{R} > \mathrm{R}_{cr}$, there can be no steady flow; when $\mathrm{R} = \mathrm{R}_{cr}$, the perturbation discontinuously reaches a non-zero amplitude, though this is still assumed so small that the expansion in powers of $|A|^2$ is valid.‡ In the range $\mathrm{R}_{cr}' < \mathrm{R} < \mathrm{R}_{cr}$, the unperturbed flow is *metastable*, being stable with respect to infinitesimal perturbations but unstable with respect to those with finite amplitude (the continuous curve; the broken curve shows the unstable branch).

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† Strictly speaking, the third-order terms give, on averaging, not zero, but fourth-order terms, which we suppose included among the fourth-order terms in the expansion.

‡ Such systems are said to have *hard* self-excitation, in contrast to those with *soft* self-excitation, which are unstable with respect to infinitesimal perturbations.

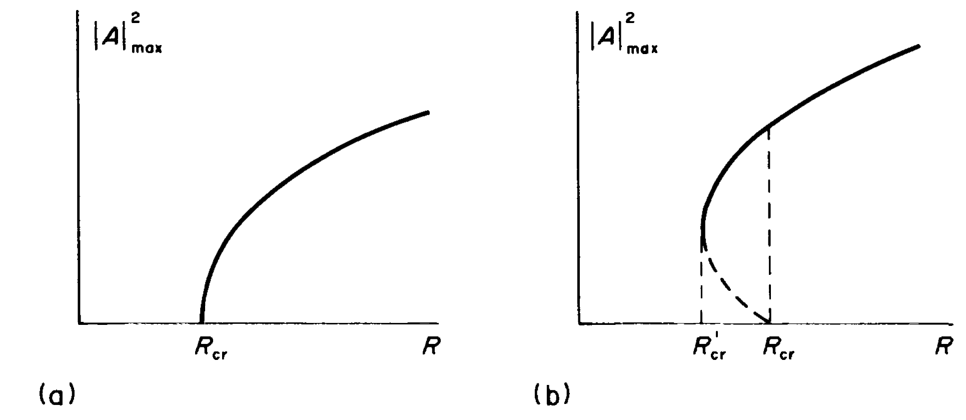

FIG. 13

Let us now return to the non-steady flow which occurs when $R > R_{cr}$, as a result of the instability with respect to small perturbations. For R close to R_{cr} the latter flow can be represented by superposing on the steady flow $\mathbf{v}_0(\mathbf{r})$ a periodic flow $\mathbf{v}_1(\mathbf{r}, t)$, with a small but finite amplitude which increases with R as in (26.10). The velocity distribution in this flow is of the form

$$\mathbf{v}_1 = \mathbf{f}(\mathbf{r})e^{-i(\omega_1 t+\beta_1)}, \tag{26.12}$$

where $\mathbf{f}$ is a complex function of the coordinates, and β_1 is some initial phase. For large $R - R_{cr}$, the separation of the velocity into $\mathbf{v}_0$ and $\mathbf{v}_1$ is no longer meaningful. We then have simply some periodic flow with frequency ω_1. If, instead of the time, we use as an independent variable the phase $\phi_1 \equiv \omega_1 t + \beta_1$, then we can say that the function $\mathbf{v}(\mathbf{r}, \phi_1)$ is a periodic function of ϕ_1, with period 2π. This function, however, is no longer a simple trigonometrical function. Its expansion in Fourier series

$$\mathbf{v} = \sum_p \mathbf{A}_p(\mathbf{r})e^{-i\phi_1 p} \tag{26.13}$$

(where the summation is over all integers p, positive and negative) includes not only terms with the fundamental frequency ω_1, but also terms whose frequencies are integral multiples of ω_1.

Equation (26.7) determines only the modulus of the time factor $A(t)$, and not its phase ϕ_1, which remains essentially indeterminate, and depends on the particular initial conditions which happen to occur at the instant when the flow begins. The initial phase β_1 can have any value, depending on these conditions. Thus the periodic flow under consideration is not uniquely determined by the given steady external conditions in which the flow takes place. One quantity—the initial phase of the velocity—remains arbitrary. We may say that the flow has one degree of freedom, whereas steady flow, which is entirely determined by the external conditions, has no degrees of freedom.

PROBLEM

Derive the equation for the energy balance between the unperturbed flow and a superimposed perturbation, without assuming that the latter is weak.

SOLUTION. Substituting (26.2) in (26.1), but not omitting the term of the second order in $\mathbf{v}_1$, we have

$$\partial\mathbf{v}_1/\partial t + (\mathbf{v}_0\cdot\mathbf{grad})\mathbf{v}_1 + (\mathbf{v}_1\cdot\mathbf{grad})\mathbf{v}_0 + (\mathbf{v}_1\cdot\mathbf{grad})\mathbf{v}_1 = -\mathbf{grad}\, p_1 + (1/R)\triangle\mathbf{v}_1; \tag{1}$$